Emmanuel College
St Andrew's Street
Cambridge CB2 3AP
Telephone: +44 (0)1223 334200
10 August 2023

Emmanuel College: Mathematics

To: First Year Mathematicians

Many congratulations on your place at Emmanuel. We are looking forward to seeing you at the start of term.

As you are probably already aware, the Mathematics course at Cambridge is different from school mathematics both in character (e.g., exercises are both more difficult and longer) and in structure (e.g., unlike at many other universities, there are no modules). Also, unlike much school mathematics, the course quite often requires significant work in order to master the subject matter: indeed, a lot of material is covered in lectures and at a high pace[1]. It is therefore important that you get your mathematical studies off to a good start. Also, mathematics is something which requires regular practice to avoid becoming rusty, and it takes a while to get back up to speed even after a break of just a few months.

In order to help you prepare for, and adjust to, Cambridge Mathematics, we have two suggestions, and a short induction session.

- First, the Faculty of Mathematics produces a workbook which has lots of exercises in it; see[2]

  https://www.maths.cam.ac.uk/undergrad/admissions/files/workbook.pdf.

  Please do not feel the need to produce a perfect solution for each of these questions, but rather sketch out solutions for yourself, making sure that you remember this sort of material from school. It may help to do a selection of questions in detail. Depending on your “year-13” syllabus (e.g. A-level, IB, etc.), you may not be familiar with all the topics covered in this booklet (for instance, hyperbolic functions). If so, then some background reading (perhaps from the internet or, in a past-life, at your local library) would help to fill in the gaps; otherwise unfamiliar material will be covered in the first few lectures of the relevant lecture course (although, as noted above, at some pace).

- Second, an example sheet is enclosed. Please work through this and write out detailed solutions to all the questions you can. The questions on this sheet are more difficult, so do not worry if you cannot do them all perfectly, but please *attempt every question*. You will have a supervision very close to the start of term (time to be advised, but probably *towards the start of your first full week of residence*), when we will go through this example sheet. Hence please do the

[1] Some of you may assume that this is yet another example of a “teacher” laying it on a little thick, and that it is safe to ignore this statement; unfortunately, some of you would be ill-advised to do so.
[2] At the time of writing the link to the *Mathematics Workbook* refers to the version for the 2022-23 academic year. By the time of reading it should refer to the 2023-24 version, which we expect to be very similar.

questions *before* you come up to Cambridge and bring your attempts to all the questions with you to the induction course (see below). Please now re-read the last sentence and note the *before*.

- Third, we will be running a short induction course to the Mathematical Tripos, on **Friday 29 September 2023**. During this, we outline how Cambridge Mathematics works, and give you a brief first sight of the sort of mathematics that you can expect to meet during your time here. There will also be an opportunity to meet your peers and some of the other mathematicians in Emmanuel. A timetable will be circulated closer to time, but the first session is likely to **start at 9:15am.**

  Beforehand you might like to peruse the booklet *Courses in Part IA* produced by the Faculty of Mathematics: see[3]

  http://www.maths.cam.ac.uk/undergrad/course/coursesIA.pdf.

Next, a practicality; given that the induction course starts at 9:15 am on Friday morning, we expect many of you will want to come up on Thursday 28 September 2023 to settle in before the start of the course. It is important therefore that you **must** email the Senior Tutor's Secretary, Mrs Marion Dorkings (email: tutorial-office@emma.cam.ac.uk), to seek permission to arrive in College early, as most other first years are not expected to arrive until Saturday 30 September (international students (with permission) may arrive earlier if necessary).

Meanwhile, we hope you enjoy the rest of the vacation.

Congratulations once again.

With best wishes

Stephen

Stephen J. Cowley

Directors of Studies in Mathematics

[3] At the time of writing the link to the booklet *Courses in Part IA* refers to the version for the 2022-23 academic year. By the time of reading it should refer to the 2023-24 version, which we expect to be very similar.

Mathematical Tripos Part IA: Introductory Examples Sheet

The material on this sheet is not particularly related to any course. The intention is to give you something to do until the lecture courses have progressed far enough for you to tackle the first proper examples sheet. Some of the questions have hard bits, so don't expect to do everything!

1 Sketch the graph of the function $y(x)$ given by

$$y(x) = \frac{x-3}{(x+1)(x-2)},$$

indicating the positions of the turning points.

Prove that there is a range of values which y cannot take if x is real.

2 (i) Find dy/dx when $\tan y = \left(\dfrac{x-1}{2-x}\right)^{\frac{1}{2}}$, where $1 < x < 2$. Hence integrate

$$\int \frac{1}{\sqrt{(x-1)(2-x)}}\,dx \qquad (1 < x < 2).$$

(ii) Guess the integral

$$\int \frac{1}{\sqrt{(t-a)(b-t)}}\,dt \qquad (a < t < b)$$

and verify your guess by differentiation.

(iii) Do the integral of part (ii) by writing $(t-a)(b-t)$ in the form $A^2 - (t-B)^2$ and using the new variable θ defined by $(t-B) = A\sin\theta$. Check using trigonometry that your answers agree.

3 A curve is given parametrically by

$$x = a(\theta - \sin\theta), \quad y(\theta) = a(1+\cos\theta),$$

where a is a positive constant. Show that the gradient of the curve is $-\cot\frac{1}{2}\theta$. Sketch the curve in the x-y plane, explaining carefully how you discovered its main features.

A small smooth bead is slides on the curve. Show that its velocity $\big(\dot{x}, \dot{y}\big)$, where the dot denotes differentiation with respect to time, t, is

$$2a\dot{\theta}\sin\tfrac{1}{2}\theta\,\Big(\sin\tfrac{1}{2}\theta,\ -\cos\tfrac{1}{2}\theta\Big).$$

Show also, by conservation of energy, that if the bead started at rest with $\theta = \theta_0$, where $0 < \theta < \pi$, then

$$\dot{\theta} = \frac{\sqrt{(g/2a)(\cos\theta_0 - \cos\theta)}}{\sin\frac{1}{2}\theta}.$$

Hence find the time for the bead to reach the point with $\theta = \pi$. Comment on your result.

4 Let

$$I_n = \int_0^\infty \operatorname{sech}^n u \;\mathrm{d}u.$$

By integrating by parts, show that for $n > 0$

$$\int_0^\infty (\operatorname{sech}^{n+2} u\ \sinh u)\sinh u\;\mathrm{d}u = (n+1)^{-1} I_n$$

and deduce that $(n+1)I_{n+2} = nI_n$. Find the value of I_6.

[NB: $\sinh x = (e^x - e^{-x})/2$, $\quad\cosh x = (e^x + e^{-x})/2$, $\quad\operatorname{sech} x = (\cosh x)^{-1}$. You will need to use the identity $\sinh^2 u = \cosh^2 u - 1$.]

5 Show that $\int_0^{2\pi} e^{in\theta} \mathrm{d}\theta = 0$ where n is a non-zero integer. Express $\cos\theta$ in terms of $e^{i\theta}$.

Write down the coefficient of the term which is independent of x in binomial expansion of $(x+x^{-1})^{2n}$, where n is a positive integer. By taking $x = e^{i\theta}$ and using the results of the first paragraph, evaluate

$$\int_0^{2\pi} \cos^{2n}\theta \,\mathrm{d}\theta \qquad \text{and} \qquad \int_0^{2\pi} \cos^{2n} 2\theta \,\mathrm{d}\theta.$$

6 A uniform wire is bent into the shape of a circular arc of radius a which subtends an angle 2α at the centre of the circle. The distance, d, of the centre of mass of the wire from the centre of the circle may be written as $d = f(\alpha)$. By cutting the arc into two similar arcs, or otherwise, show that

$$f(\alpha) = f(\alpha/2)\cos(\alpha/2). \qquad (*)$$

Show that $f(\alpha) = (A\sin\alpha)/\alpha$ satisfies this equation. Assume that this is the correct form for $f(\alpha)$. By considering the case when α is very small, show that $d = (a\sin\alpha)/\alpha$. Show that the corresponding result for a lamina in the shape of a sector of a circle is $d = (2a\sin\alpha)/(3\alpha)$.

Try solving the functional equation $(*)$ by considering the function $g(x)$ defined by $g(x) = \dfrac{\sin x}{x f(x)}$.

7 (i) Show that $\displaystyle\int_0^1 \frac{\sin^{-1} x}{\sqrt{1-x^2}}\, dx = \frac{\pi^2}{8}$.

(ii) By writing $\displaystyle\sin^{-1} x = \int_0^x \frac{dy}{\sqrt{1-y^2}}$, show that $\displaystyle\sin^{-1} x = \sum_0^\infty \frac{(2n)!}{2^{2n}(n!)^2}\frac{x^{2n+1}}{2n+1}$.

(iii) Let $\displaystyle I_n = \int_0^1 \frac{x^{2n+1}}{\sqrt{1-x^2}}\, dx$. Show that $\displaystyle I_n = \frac{2n}{2n+1} I_{n-1}$ and hence that $\displaystyle I_n = \frac{2^{2n}(n!)^2}{(2n+1)!}$.

Deduce that $\displaystyle\sum_0^\infty \frac{1}{(2n+1)^2} = \frac{\pi^2}{8}$.

Noting that $\displaystyle 4\sum_1^\infty \frac{1}{(2n)^2} = \sum_1^\infty \frac{1}{n^2}$, deduce also that $\displaystyle\zeta(2) = \frac{\pi^2}{6}$, where $\displaystyle\zeta(k) = \sum_{n=1}^\infty \frac{1}{n^k}$.

8 For any given polynomial f, the factor theorem states that $f(a) = 0$ if and only if $f(x) = (x-a)g(x)$ where $g(x)$ is a polynomial. By comparison with the factor theorem, show that the result

$$\sin\pi x = \pi x\left(1-\frac{x^2}{1^2}\right)\left(1-\frac{x^2}{2^2}\right)\left(1-\frac{x^2}{3^2}\right)\cdots \qquad (*)$$

is plausible. Can $(*)$ be derived from the factor theorem?

(i) Use $(*)$ to derive the Wallis's formula

$$\frac{\pi}{2} = \frac{2}{1}\times\frac{2}{3}\times\frac{4}{3}\times\frac{4}{5}\times\frac{6}{5}\times\frac{6}{7}\times\frac{8}{7}\times\frac{8}{9}\times\cdots .$$

Use $(*)$ and the double angle formula for $\sin 2\pi x$ to obtain an infinite product for $\cos\pi x$ and hence obtain an infinite product for $\sqrt{2}$.

(ii) By considering the Taylor series for $\sin\pi x$, obtain the expression in the previous question for $\zeta(2)$. Obtain the corresponding expression for $\zeta(4)$.

9 The aim of this question is to prove that π is irrational by contradiction. Suppose $\pi = p/q$ where p and q are positive integers and let

$$I_n = \int_0^{\pi} \mathrm{f}(x) \sin x \, \mathrm{d}x \,,$$

where $\mathrm{f}(x) = \dfrac{q^n x^n (\pi - x)^n}{n!}$ and q is a positive integer.

(i) Show that

$$I_n = \sum_{j=0}^{n} (-1)^j \left(\mathrm{f}^{(2j)}(\pi) + \mathrm{f}^{(2j)}(0) \right)$$

where $\mathrm{f}^k(x)$ denotes the kth derivative of $\mathrm{f}(x)$. Deduce that I_n is an integer.

(ii) Show that

$$I_n \leq \frac{\pi}{n!} \left(\frac{q\pi^2}{4} \right)^n ,$$

and deduce that $I_n \to 0$ as $n \to \infty$.

(iii) Deduce that π is irrational.